

# On Various Parameters of $\mathbb{Z}_q$ -Simplex codes for an even integer $q$ \*

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**Proposed running head:** On Various Parameters of  $\mathbb{Z}_q$ -Simplex codes for an even integer  $q$

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### **Abstract**

In this paper, we defined the  $\mathbb{Z}_q$ -linear codes and discussed its various parameters. We constructed  $\mathbb{Z}_q$ -Simplex code and  $\mathbb{Z}_q$ -MacDonald code and found its parameters. We have given a lower and an upper bounds of its covering radius for  $q$  is an even integer.

*Keywords:* Codes over finite rings,  $\mathbb{Z}_q$ -linear code,  $\mathbb{Z}_q$ -Simplex code,  $\mathbb{Z}_q$ -MacDonald code, Covering radius.

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# 1 Introduction

A code  $C$  is a subset of  $\mathbb{Z}_q^n$ , where  $\mathbb{Z}_q$  is the set of integer modulo  $q$  and  $n$  is any positive integer. Let  $x, y \in \mathbb{Z}_q^n$ , then the distance between  $x$  and  $y$  is the number of coordinates in which they differ. It is denoted by  $d(x, y)$ . Clearly  $d(x, y) = wt(x - y)$ , the number of non-zero coordinates in  $x - y$ .  $wt(x)$  is called *weight of  $x$* . The minimum distance  $d$  of  $C$  is defined by

$$d = \min\{d(x, y) \mid x, y \in C \text{ and } x \neq y\}.$$

The minimum weight of  $C$  is  $\min\{wt(c) \mid c \in C \text{ and } c \neq 0\}$ . A code of length  $n$  cardinality  $M$  with minimum distance  $d$  over  $\mathbb{Z}_q$  is called  $(n, M, d)$   $q$ -ary code. For basic results on coding theory, we refer [16].

We know that  $\mathbb{Z}_q$  is a group under addition modulo  $q$ . Then  $\mathbb{Z}_q^n$  is a group under coordinatewise addition modulo  $q$ . A subset  $C$  of  $\mathbb{Z}_q^n$  is said to be a  $q$ -ary code. If  $C$  is a subgroup of  $\mathbb{Z}_q^n$ , then  $C$  is called a  $\mathbb{Z}_q$ -linear code. Some authors are called this code as *modular code* because  $\mathbb{Z}_q^n$  is a module over the ring  $\mathbb{Z}_q$ . In fact, it is a free  $\mathbb{Z}_q$ -module. Since  $\mathbb{Z}_q^n$  is a free  $\mathbb{Z}_q$ -module, it has a basis. Therefore, every  $\mathbb{Z}_q$ -linear code has a basis. Since  $\mathbb{Z}_q$  is finite, it is finite dimension.

Every  $k$  dimension  $\mathbb{Z}_q$ -linear code with length  $n$  and minimum distance  $d$  is called  $[n, k, d]$   $\mathbb{Z}_q$ -linear code. A matrix whose rows are a basis elements of the  $\mathbb{Z}_q$ -linear code is called a *generator matrix* of  $C$ . There are many researchers doing research on code over finite rings [4], [9], [10], [11], [13], [14], [18]. In the last decade, there are many researchers doing research on codes over  $\mathbb{Z}_4$  [1], [2], [3], [8], [15].

In this correspondence, we concentrate on code over  $\mathbb{Z}_q$  where  $q$  is even. We constructed some new codes and obtained its various parameters and its covering radius. In particular, we defined  $\mathbb{Z}_q$ -Simplex code,  $\mathbb{Z}_q$ -MacDonald code and studied its various parameters. Section 2 contains basic results for the  $\mathbb{Z}_q$ -linear codes and we constructed some  $\mathbb{Z}_q$ -linear code and given its parameters.  $\mathbb{Z}_q$ -Simplex code and perfect code are given in section 3 and finally, section 4 we determined the covering radius of these codes and  $\mathbb{Z}_q$ -MacDonald code.

## 2 $\mathbb{Z}_q$ -Linear code

Let  $C$  be a  $\mathbb{Z}_q$ -linear code. If  $x, y \in C$ , then  $x - y \in C$ . Let us consider the minimum distance of  $C$  is  $d = \min\{d(x, y) \mid x, y \in C \text{ and } x \neq y\}$ . Then

$$d = \min\{wt(x - y) \mid x, y \in C \text{ and } x \neq y\}.$$

Since  $C$  is  $\mathbb{Z}_q$ -linear code and  $x, y \in C$ ,  $x - y \in C$ . Since  $x \neq y$ ,  $\min\{wt(x - y) \mid x, y \in C \text{ and } x \neq y\} = \min\{wt(c) \mid c \in C \text{ and } c \neq 0\}$ . Thus, we have

**Lemma 2.1.** *In a  $\mathbb{Z}_q$ -linear code, the minimum distance is the same as the minimum weight.*

Let  $q$  be an even integer and let  $x, y \in \mathbb{Z}_q^n$  such that  $x_i, y_i \in \{0, \frac{q}{2}\}$ , then  $x_i \pm y_i \in \{0, \frac{q}{2}\}$ .

**Lemma 2.2.** *Let  $q$  be an integer even. If  $x, y \in \mathbb{Z}_q^n$  such that  $x_i, y_i \in \{0, \frac{q}{2}\}$ , then the coordinates of  $x \pm y$  are either 0 or  $\frac{q}{2}$ .*

Now, we construct a new code and discuss its parameters. Let  $C$  be an  $[n, k, d]$   $\mathbb{Z}_q$ -linear code. Define

$$D = \{c0c \cdots c + \alpha(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1}) \mid \alpha \in \mathbb{Z}_q, c \in C \text{ and } \mathbf{i} = ii \cdots i \in \mathbb{Z}_q^n\}.$$

Then,  $D = \{c0c \cdots c, c0c \cdots c + \mathbf{0112} \cdots \mathbf{q} - \mathbf{1}, c0c \cdots c + 2(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1}), \dots, c0c \cdots c + (q-1)(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1}) \mid c \in C \text{ and } \mathbf{i} \in \mathbb{Z}_q^n\}$ . Since any  $\mathbb{Z}_q$ -linear combination of  $D$  is again an element in  $D$ , therefore the minimum distance of  $D$  is  $d(D) = \min\{wt(c0c \cdots c), wt(c0c \cdots c + \mathbf{0112} \cdots \mathbf{q} - \mathbf{1}), wt(c0c \cdots c + 2(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1})), \dots, wt(c0c \cdots c + (q-1)(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1})) \mid c \in C \text{ and } \mathbf{i} \in \mathbb{Z}_q^n\}$ .

Clearly  $\min\{wt(c0c \cdots c) \mid c \in C \text{ and } c \neq 0\} \geq qd$ .

Let  $c \in C$ . Let us take  $c$  has  $r_i$   $i$ 's where  $i = 0, 1, 2, \dots, q-1$ . Then for  $1 \leq i \leq q-1$ ,

$$wt(c + \mathbf{i}) = \sum_{j=0}^{q-1} r_j - r_{q-i}.$$

That is  $wt(c + \mathbf{i}) = n - r_{q-i}$ . Therefore

$$\begin{aligned} wt(c0c \cdots c + \mathbf{0112} \cdots \mathbf{q} - \mathbf{1}) &= wt(c + \mathbf{0}) + 1 + wt(c + \mathbf{1}) + wt(c + \mathbf{2}) \\ &\quad + \cdots + wt(c + \mathbf{q} - \mathbf{1}) \\ &= n - r_0 + 1 + n - r_{q-1} + n - r_{q-2} + \cdots \\ &\quad + n - r_1 \\ &= (q-1)n + 1 \end{aligned}$$

Similarly, for every integer  $i$  which is relatively prime to  $q$

$$wt((c0c \cdots c) + i(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1})) = (q-1)n + 1$$

For other  $i$ 's

$$\begin{aligned}
\min_{i \in \mathbb{Z}_q} \{wt(c0c \cdots c + i(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1}))\} &= wt(c + \mathbf{0}) + 1 \\
&\quad + wt(c \cdots c + \frac{q}{2}(\mathbf{12} \cdots \mathbf{q} - \mathbf{1})) \\
&= wt(c + \mathbf{0}) + 1 \\
&\quad + wt(c \cdots c + (\frac{\mathbf{q}}{2}\mathbf{0}\frac{\mathbf{q}}{2}\mathbf{0} \cdots \frac{\mathbf{q}}{2}\mathbf{0}\frac{\mathbf{q}}{2})) \\
&= \frac{q}{2}wt(c + \mathbf{0}) + 1 + \frac{q}{2}wt(c + \frac{\mathbf{q}}{2}) \\
&= \frac{q}{2}(n - r_0) + 1 + \frac{q}{2}(n - r_{\frac{q}{2}}) \\
\min_{i \in \mathbb{Z}_q} \{wt(c0c \cdots c + i(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1}))\} &= \frac{q}{2}n + 1 + \frac{q}{2}(n - r_0 - r_{\frac{q}{2}})
\end{aligned}$$

Hence,  $d(D) = \min\{qd, (q-1)n + 1, \frac{q}{2}n + 1 + \frac{q}{2}(n - r_0 - r_{\frac{q}{2}})\}$ . Thus, we have

**Theorem 2.3.** *Let  $C$  be an  $[n, k, d]$   $\mathbb{Z}_q$ -linear code, then the*

$D = \{c0c \cdots c + \alpha(\mathbf{0112} \cdots \mathbf{q} - \mathbf{1}) \mid \alpha \in \mathbb{Z}_q, c \in C \text{ and } \mathbf{i} = ii \cdots i \in \mathbb{Z}_q^n\}$  *is a  $[qn + 1, k + 1, d(D)]$   $\mathbb{Z}_q$ -linear code.*

If there is a codeword  $c \in C$  such that it has only 0 and  $\frac{q}{2}$  as coordinates, then

$$\begin{aligned}
wt(c0c \cdots c + \mathbf{0}\frac{\mathbf{q}}{2}\mathbf{0}\frac{\mathbf{q}}{2}\mathbf{0}\frac{\mathbf{q}}{2} \cdots \mathbf{0}\frac{\mathbf{q}}{2}) &= wt(c + 0) + 1 + wt(c + \frac{q}{2}) + wt(c \\
&\quad + 0) + \cdots + w(c + \frac{q}{2}) \\
&= 1 + r_{\frac{q}{2}} + r_0 + r_{\frac{q}{2}} + \cdots + r_0 \\
&= \frac{q}{2}(r_0 + r_{\frac{q}{2}}) + 1 \\
wt(c0c \cdots c + \mathbf{0}\frac{\mathbf{q}}{2}\mathbf{0}\frac{\mathbf{q}}{2}\mathbf{0}\frac{\mathbf{q}}{2} \cdots \mathbf{0}\frac{\mathbf{q}}{2}) &= \frac{q}{2}n + 1.
\end{aligned}$$

Hence,  $d(D) = \min\{qd, \frac{q}{2}n + 1\}$ . Thus, we have

**Corollary 2.4.** *If there is a  $c \in C$  such that  $c_i = 0$  or  $\frac{q}{2}$  and if  $n \leq 2d - 1$ , then  $d(D) = \frac{q}{2}n + 1$ .*

### 3 $\mathbb{Z}_q$ -Simplex codes

Let  $G$  be a matrix over  $\mathbb{Z}_q$  whose columns are one non-zero element from each 1-dimensional submodule of  $\mathbb{Z}_q^2$ . Then this matrix is equivalent to

$$G_2 = \left[ \begin{array}{c|c|cccc} 0 & 1 & 1 & 2 & \cdots & q-1 \\ \hline 1 & 0 & 1 & 1 & \cdots & 1 \end{array} \right].$$

Clearly  $G_2$  generates  $[q+1, 2, \frac{q}{2}+1]$  code. Inductively, we define

$$G_{k+1} = \left[ \begin{array}{c|c|c|c|c|c} 00 \cdots 0 & 1 & 11 \cdots 1 & 22 \cdots 2 & \cdots & q-1q-1 \cdots q-1 \\ \hline & 0 & & & & \\ & \vdots & G_k & G_k & \cdots & \\ & 0 & & & & G_k \end{array} \right]$$

for  $k \geq 2$ . Clearly this  $G_{k+1}$  matrix generates  $[n_{k+1} = \frac{q^{k+1}-1}{q-1}, k+1, d]$  code. We call this code as  $\mathbb{Z}_q$ -Simplex code. This type of k-dimensional code is denoted by  $S_k(q)$ . For simplicity, we denote it by  $S_k$ .

**Theorem 3.1.**  $S_k(q)$  is  $[n_k = \frac{q^k-1}{q-1}, k, \frac{q}{2}n_{k-1}+1]$  code.

*Proof.* We prove this theorem by induction on k. For  $k=2$ , from the generator matrix  $G_2$ , it is clear that  $d = \frac{q}{2}+1$  and the theorem is true. Since there is a codeword  $c = 0\frac{q}{2}0\frac{q}{2}\cdots 0\frac{q}{2}0\frac{q}{2} \in S_2$  and  $n = q+1 \leq 2(\frac{q}{2}+1) - 1 = 2d-1$ , by Corollary 2.4 implies  $d(S_3) = \frac{q}{2}n_2+1$  and hence the  $S_3$  is  $[n_3 = \frac{q^3-1}{q-1}, 3, \frac{q}{2}n_2+1]$  code. Since  $c0c \cdots c + \frac{q}{2}(\mathbf{0112} \cdots \mathbf{q-1}) \in S_3$  whose coordinates are either 0 or  $\frac{q}{2}$  and satisfies the conditions of the Corollary 2.4, therefore  $d(S_4) = \frac{q}{2}n_3+1$  and hence the  $S_4$  is  $[n_4 = \frac{q^4-1}{q-1}, 4, \frac{q}{2}n_3+1]$  code. By induction we can assume that this theorem is true for all less than k. That is, there is a code  $c \in S_{k-1}$  whose coordinates are either 0 or  $\frac{q}{2}$  and  $n_{k-1} \leq 2d_{k-1} - 1$ . By Corollary 2.4,  $d_k = \frac{q}{2}n_{k-1}+1$ . Therefore  $S_k(q)$  is an  $[\frac{q^k-1}{q-1}, k, \frac{q}{2}n_{k-1}+1]$   $\mathbb{Z}_q$ -linear code. Thus we proved.  $\square$

This code seems to be a Simplex code over finite field but not because the parameters differ. From the matrix  $G_k$ , no two columns are linearly dependent. Therefore the minimum distance of its dual is greater than or equal to 3. Since in the first block of the matrix  $G_k$ , there are three columns whose transpose matrix are  $(0, 0, \cdots, 0, 1)$ ,  $(0, 0, \cdots, 0, 1, 0)$ ,  $(0, 0, \cdots, 0, 1, 1)$ . These are linearly dependent. Therefore, the minimum distance of the dual code is less than or equal to 3. Hence the dual of  $S_k$  is  $[n_k = \frac{q^k-1}{q-1}, n_k - k, 3]$   $\mathbb{Z}_q$ -linear code. One can check easily that the spheres of radius 1 around codewords cover the whole space  $\mathbb{Z}_q^n$  and the spheres are disjoint. Hence the dual of this code is perfect. This is, this code is equivalent to the q-ary Hamming codes because any code with these parameters  $(n = \frac{q^k-1}{q-1}, q^{n-k}, 3)$  is equivalent to the q-ary Hamming code[16].

## 4 Covering radius

The *covering radius* of a code  $C$  over  $\mathbb{Z}_q$  with respect to the Hamming distance  $d$  is given by

$$R(C) = \max_{u \in \mathbb{Z}_q^n} \left\{ \min_{c \in C} \{d(u, c)\} \right\}.$$

It is easy to see that  $R(C)$  is the least positive integer  $r$  such that

$$\mathbb{Z}_q^n = \cup_{c \in C} S_r(c)$$

where

$$S_r(u) = \{v \in \mathbb{Z}_q^n \mid d(u, v) \leq r\}$$

for any  $u \in \mathbb{Z}_q^n$ .

**Proposition 4.1.** [5] *If appending( puncturing)  $r$  number of columns in a code  $C$ , then the covering radius of  $C$  is increased( decreased ) by  $r$ .*

**Proposition 4.2.** [17] *If  $C_0$  and  $C_1$  are codes over  $\mathbb{Z}_q^n$  generated by matrices  $G_0$  and  $G_1$  respectively and if  $C$  is the code generated by*

$$G = \left( \begin{array}{c|c} 0 & G_1 \\ \hline G_0 & A \end{array} \right),$$

*then  $r(C) \leq r(C_0) + r(C_1)$  and the covering radius of  $C$  satisfy the following*

$$r(C) \geq r(C_0) + r(C_1).$$

*Since the covering radius of  $C$  generated by*

$$G = \left( \begin{array}{c|c} 0 & G_1 \\ \hline G_0 & A \end{array} \right),$$

*is greater than or equal to  $r(C_0) + r(C')$  where  $C_0$  and  $C'$  are codes generated by  $\begin{bmatrix} 0 \\ G_0 \end{bmatrix} = \begin{bmatrix} G_1 \\ A \end{bmatrix}$  and  $\begin{bmatrix} G_1 \\ A \end{bmatrix}$ , respectively, this implies  $r(C) \geq r(C_0) + r(C_1)$  because  $C_1$  is a subcode of the code  $C'$ .*

A  $q$ -ary repetition code  $C$  over a finite field  $\mathbb{F}_q$  with  $q$  elements is an  $[n, 1, n]$  linear code. The covering radius of  $C$  is  $\lfloor \frac{n(q-1)}{q} \rfloor$  [12]. For basic results on covering radius, we refer to [5], [6]. Now, we consider the repetition code over  $\mathbb{Z}_q$ . There are two types of repetition codes.

Type I. Unit repetition code generated by  $G_u = [\overbrace{uu \dots u}^n]$  where  $u$  is a unit element of  $\mathbb{Z}_q$ . This matrix generates  $C_u$  is  $[n, 1, n]$   $\mathbb{Z}_q$ -linear code. That is,  $C_u$  is  $(n, q, n)$   $q$ -ary repetition code. We call this as *unit repetition code*.

Type II. Zero divisor repetition code is generated by the matrix

$G_v = [\overbrace{vv \dots v}^n]$  where  $v$  is a zero divisor in  $\mathbb{Z}_q$ . That is,  $v$  is not a relatively prime to  $q$ . This is an  $(n, \frac{q}{v}, n)$  code over  $\mathbb{Z}_q$ . This code is denoted by  $C_v$ . This code is called *zero divisor repetition code*.

With respect to the Hamming distance the covering radius of  $C_1$  is  $\lfloor \frac{n(q-1)}{q} \rfloor$  [12] but clearly the covering radius of  $C_v$  is  $n$  because code symbols appear in this code are zero divisors only. Thus, we have

**Theorem 4.3.**  $R(C_v) = n$  and  $R(C_u) = \lfloor \frac{(q-1)n}{q} \rfloor$ .

Let  $\phi(q) = \#\{i \mid 1 \leq i < q \text{ \& } (i, q) = 1\}$  be the Euler  $\phi$ -function. Let  $U = \{i \in \mathbb{Z} \mid 1 \leq i < q \text{ \& } (i, q) = 1\}$  be the set of all units in  $\mathbb{Z}_q$  and let  $O = \mathbb{Z}_q \setminus U$  be the set which contains all zero divisors and 0. Let  $C$  be a  $\mathbb{Z}_q$ -linear code generated by the matrix

$$[\overbrace{11 \dots 1}^n \overbrace{22 \dots 2}^n \dots \overbrace{q-1q-1 \dots q-1}^n],$$

then this code is equivalent to a code whose generator matrix is  $[u_1 u_1 \dots$

$$u_1 u_2 u_2 \dots u_2 \dots u_{\phi(q)} u_{\phi(q)} \dots u_{\phi(q)} o_1 o_1 \dots o_1 o_2 o_2 \dots o_2 \dots o_r o_r \dots o_r]$$

where  $r = q - 1 - \phi(q)$ . Let  $A$  be a code equivalent to the unit repetition code of length

$$\phi(q)n \text{ generated by } [u_1 u_1 \dots u_1 u_2 u_2 \dots u_2 \dots u_{\phi(q)} u_{\phi(q)} \dots$$

$u_{\phi(q)}]$ , then by the above theorem,  $R(A) = \lfloor \frac{(q-1)\phi(q)n}{q} \rfloor$ . Let  $B$  be a code equivalent to the

zero divisor repetition code of length  $(q-1-\phi(q))n$  generated by  $[o_1 o_1 \dots o_1 o_2 o_2 \dots o_2 \dots o_r o_r \dots o_r]$ ,

then by the above theorem,  $R(B) = (q-1-\phi(q))n$ . By Proposition 4.2,  $R(C) \geq$

$$\lfloor \frac{(q-1)\phi(q)n}{q} \rfloor + (q-1-\phi(q))n.$$

Without loss of generality we can assume that the generator matrix of  $A$  as  $[111 \dots 1]$ .

Since  $R(A) = \lfloor \frac{(q-1)\phi(q)n}{q} \rfloor$  and  $C$  is obtained by appending some  $(q-1-\phi(q))n$  columns

to  $A$ , by Proposition 4.1 the covering radius of  $C$  is increased by at most  $(q-1-\phi(q))n$ .

Therefore,  $R(C) \leq \lfloor \frac{(q-1)\phi(q)n}{q} \rfloor + (q-1-\phi(q))n$ . Thus, we have

**Theorem 4.4.** Let  $C$  be a  $\mathbb{Z}_q$ -linear code generated by the matrix  $[\overbrace{11 \dots 1}^n$

$$\overbrace{22 \dots 2}^n \dots \overbrace{q-1q-1 \dots q-1}^n]. \text{ Then } C \text{ is a } [(q-1)n, 1, \frac{q}{2}n] \mathbb{Z}_q\text{-linear code with } R(C) = \lfloor \frac{(q-1)\phi(q)n}{q} \rfloor + (q-1-\phi(q))n.$$

Now, we see the covering radius of  $\mathbb{Z}_q$ -Simplex code. The covering radius of Simplex codes and MacDonald codes over finite field and finite rings were discussed in [12], [14].

**Theorem 4.5.** For  $k \geq 2$ ,  $R(S_{k+1}) \leq \frac{(k-1)(q-1)\phi(q) + (q^2 - q - \phi(q))(q^{k+1} - q^2)}{q(q-1)^2} + R(S_2)$



*Proof.* For  $k \geq 2$ ,  $S_{k+1}$  is  $[n_{k+1} = \frac{q^{k+1}-1}{q-1}, k+1, \frac{q}{2}n_k + 1]$   $\mathbb{Z}_q$ -linear code. By Proposition 4.2 and Theorem 4.4 give

$$\begin{aligned}
R(S_{k+1}) &\leq (1 + \lfloor \frac{(q-1)\phi(q)n_k}{q} \rfloor + (q-1-\phi(q))n_k) + R(S_k) \\
&\leq (1 + \frac{(q-1)\phi(q)n_k}{q} + (q-1-\phi(q))n_k) + R(S_k) \\
&\leq 1 + \frac{q^2 - q - \phi(q)}{q}n_k + R(S_k) \\
R(S_{k+1}) &\leq (1 + \frac{q^2 - q - \phi(q)}{q}n_k) + R(S_k)
\end{aligned}$$

This implies

$$R(S_k) \leq (1 + \frac{q^2 - q - \phi(q)}{q}n_{k-1}) + R(S_{k-1}).$$

Combining these two, we get

$$R(S_{k+1}) \leq (1 + \frac{q^2 - q - \phi(q)}{q}n_k) + (1 + \frac{q^2 - q - \phi(q)}{q}n_{k-1}) + R(S_{k-1})$$

Similarly, if we continue, we get

$$\begin{aligned}
R(S_{k+1}) &\leq (1 + \frac{q^2 - q - \phi(q)}{q}n_k) + (1 + \frac{q^2 - q - \phi(q)}{q}n_{k-1}) + \dots + \\
&\quad (1 + \frac{q^2 - q - \phi(q)}{q}n_2) + R(S_2)
\end{aligned}$$

Since  $n_k = \frac{q^k-1}{q-1}$ , for  $k \geq 2$ , therefore

$$\begin{aligned}
R(S_{k+1}) &\leq (k-1) + \frac{q^2 - q - \phi(q)}{q} \left( \frac{q^k - 1}{q-1} + \frac{q^{k-1} - 1}{q-1} + \dots + \frac{q^2 - 1}{q-1} \right) + R(S_2) \\
&\leq (k-1) + \frac{q^2 - q - \phi(q)}{q} \left( \frac{q^k + q^{k-1} + \dots + q^2 - (k-1)}{q-1} \right) + R(S_2) \\
&\leq \frac{(k-1)\phi(q) + (q^2 - q - \phi(q))((q^{k+1} - 1)/(q-1) - (q+1))}{q(q-1)} + R(S_2) \\
R(S_{k+1}) &\leq \frac{(k-1)(q-1)\phi(q) + (q^2 - q - \phi(q))(q^{k+1} - q^2)}{q(q-1)^2} + R(S_2)
\end{aligned}$$

Hence proved.  $\square$

In particular, for  $q = 4$ ,  $R(S_{k+1}) \leq \frac{5 \cdot 4^{k+1} + 3k - 29}{18}$  for  $k \geq 2$  because of simple calculation  $R(S_2) = 3$ .

Now, we can define a new code which is similar to the  $\mathbb{Z}_q$ -MacDonald code. Let

$$G_{k,u} = \left( G_k \setminus \begin{pmatrix} 0 \\ G_u \end{pmatrix} \right)$$

for  $2 \leq u \leq k-1$  where  $0$  is a  $(k-u) \times \frac{q^u-1}{q-1}$  zero matrix and  $(A \setminus B)$  is a matrix obtained from the matrix  $A$  by removing the matrix  $B$ . The code generated by  $G_{k,u}$  is called  $\mathbb{Z}_q$ -MacDonald code. It is denoted by  $M_{k,u}$ . The Quaternary MacDonald codes were discussed in [7].

**Theorem 4.6.** For  $k \geq 2$  and  $0 \leq u \leq k$ ,

$$R(M_{k+1,u}) \leq \frac{(k-r+1)(q-1)\phi(q) + (q^2 - q - \phi(q))q^r(q^{k-r+1} - 1)}{q(q-1)^2} \\ + R(M_{r,u}), \text{ for } u \leq r \leq k.$$

*Proof.* By using, Proposition 4.2, we get

$$\begin{aligned} R(M_{k+1,u}) &\leq (1 + \lfloor \frac{(q-1)\phi(q)n_k}{q} \rfloor + (q-1-\phi(q))n_k) + R(M_{k,u}) \\ &\leq (1 + \frac{(q-1)\phi(q)n_k}{q} + (q-1-\phi(q))n_k) + R(M_{k,u}) \\ &\leq 1 + \frac{q^2 - q - \phi(q)}{q}n_k + R(M_{k,u}) \\ R(M_{k+1,u}) &\leq (1 + \frac{q^2 - q - \phi(q)}{q}n_k) + R(M_{k,u}) \end{aligned}$$

This implies  $R(M_{k,u}) \leq (1 + \frac{q^2 - q - \phi(q)}{q}n_{k-1}) + R(M_{k-1,u})$ . Combining these two, we get

$$R(M_{k+1,u}) \leq (1 + \frac{q^2 - q - \phi(q)}{q}n_k) + (1 + \frac{q^2 - q - \phi(q)}{q}n_{k-1}) + R(M_{k-1,u})$$

Similarly, if we continue, we get

$$\begin{aligned} R(M_{k+1,u}) &\leq (1 + \frac{q^2 - q - \phi(q)}{q}n_k) + (1 + \frac{q^2 - q - \phi(q)}{q}n_{k-1}) \\ &\quad + \cdots + (1 + \frac{q^2 - q - \phi(q)}{q}n_r) + R(M_{r,u}). \end{aligned}$$

Since  $n_k = \frac{q^k - 1}{q - 1}$ , for  $k \geq 2$ , therefore

$$\begin{aligned}
R(M_{k+1,u}) &\leq (k - r + 1) + \frac{q^2 - q - \phi(q)}{q} \left( \frac{q^k - 1}{q - 1} + \frac{q^{k-1} - 1}{q - 1} + \dots + \frac{q^r - 1}{q - 1} \right) \\
&\quad + R(M_{r,u}) \\
&\leq (k - r + 1) + \frac{q^2 - q - \phi(q)}{q} \left( \frac{q^k + q^{k-1} + \dots + q^r - (k - r + 1)}{q - 1} \right) \\
&\quad + R(M_{r,u}) \\
&\leq \frac{(k - r + 1)\phi(q) + (q^2 - q - \phi(q))q^r(q^{k-r} + q^{k-r-1} + \dots + 1)}{q(q - 1)} \\
&\quad + R(M_{r,u}) \\
&\leq \frac{(k - r + 1)\phi(q) + (q^2 - q - \phi(q))q^r(q^{k-r+1} - 1)/(q - 1)}{q(q - 1)} \\
&\quad + R(M_{r,u}) \\
R(M_{k+1,u}) &\leq \frac{(k - r + 1)(q - 1)\phi(q) + (q^2 - q - \phi(q))q^r(q^{k-r+1} - 1)}{q(q - 1)^2} \\
&\quad + R(M_{r,u}).
\end{aligned}$$

□

If  $u = k$ , then

$$R(M_{k+1,k}) \leq \lfloor \frac{(q - 1)\phi(q)n_k}{q} \rfloor + (q - 1 - \phi(q))n_k + 1 \text{ for } k \geq 2.$$

In the above theorem, if we replace  $r$  by  $u + 1$ , we get

$$\begin{aligned}
R(M_{k+1,u}) &\leq \frac{(k - u)(q - 1)\phi(q) + (q^2 - q - \phi(q))q^{u+1}(q^{k-u} - 1)}{q(q - 1)^2} \\
&\quad + \frac{(q - 1)\phi(q)n_u}{q} + (q - 1 - \phi(q))n_u + 1 \text{ for } u \geq 2.
\end{aligned}$$

Thus, we have

**Corollary 4.7.** For  $k \geq 2$  and  $k \geq u \geq 2$ ,

$$\begin{aligned}
R(M_{k+1,u}) &\leq \frac{(k - u)(q - 1)\phi(q) + (q^2 - q - \phi(q))q^{u+1}(q^{k-u} - 1)}{q(q - 1)^2} \\
&\quad + \frac{(q - 1)\phi(q)n_u}{q} + (q - 1 - \phi(q))n_u + 1.
\end{aligned}$$

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